SOLUTIONS TO FINAL EXAM

1. [12 Marks] Consider the table of values below for three rules f, g, h that assign numbers to the numbers in the interval $[0,4]$. Use only the data from this table to answer the following questions. JUSTIFY YOUR ANSWERS.

x	0	1	2	3	4
$f(x)$	3	5	-2	-1	4
$g(x)$	2	3	1	6	3
$h(x)$	±1	-4	-2	1	2

(a) Which of f, g, h is definitely not a function on $[0,4]$?

Solution h can not be a function because because $h(0)$ is both -1 and $+1$. By its definition, a function can not assign two different values to one number.

(b) Which of f, g, h can be a function which is **not** one–to–one on $[0,4]$?

Solution Since $g(1)=g(4)=3$, the function g is not one–to–one.

(c) Assume f, g, h are continuous functions on $[1,2]$. Which of f, g, h must have a root r with $1<r<2$?

Solution Since f is continuous on $[1,2]$, $f(1)=5$, $f(2)=-2$ and $5>0>-2$, the Intermediate Value Theorem says that there is some $r\in(1,2)$ such that $f(r)=0$.

(d) Assume f, g, h are continuous functions on $[1,4]$. Which of f, g, h must have a critical point in $(1,4)$?

Solution If g is not differentiable at some number $c\in(1,4)$, then $x=c$ is a critical point of g. If g is differentiable on $(1,4)$, we have $g(1)=g(4)=3$. By Rolle's Theorem, there is $c\in(1,4)$ such that $g'(c)=0$. Then $x=c$ is a critical point of g.

2. [10 Marks] **(a)** Why does the polynomial $P(x)=x^3-3x+1$ have a root between zero and one?

Solution Observe that $P(1)=-1$ and $P(0)=1$. Since $P(x)$ is continuous on $[0,1]$ and $-1<0<1$, the Intermediate Value Theorem says that $P(x)$ has a root between 0 and 1.

(b) Use the bisection method once to find an approximation of a root of $P(x)=x^3-3x+1$ between zero and one.

Solution $P(0)=1$, $P(1/2)=1/8-3/2+1=-3/8$ and $P(1)=-1$. By the Intermediate Value Theorem, $P(x)$ has a root between 0 and 1/2.

3. [36 Marks] Use any method to evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to0}\frac{x-\sin x}{\arctan x-x}$

Solution Substitution of $x=0$ produces $\frac{0}{0}$. Hence we apply L'Hopital's Rule.

$$\begin{aligned}\lim_{x\to0}\frac{x-\sin x}{\arctan x-x} &= \lim_{x\to0}\frac{D(x-\sin x)}{D(\arctan x-x)} = \lim_{x\to0}\frac{1-\cos x}{\frac{1}{1+x^2}-1}\cdot\frac{1+x^2}{1+x^2}\\ &= \lim_{x\to0}\frac{(1+x^2)(1-\cos x)}{1-(1+x^2)} = -\lim_{x\to0}\frac{(1+x^2)(1-\cos x)}{x^2}\end{aligned}$$

Substitution of $x=0$ in the latter limit produces $\frac{(1)(1-1)}{0}=\frac{0}{0}$. Hence we apply L'Hopital's Rule again.

$$\begin{aligned}\lim_{x\to0}\frac{x-\sin x}{\arctan x-x} &= -\lim_{x\to0}\frac{D\left[(1+x^2)(1-\cos x)\right]}{D\left(x^2\right)}\\ &= -\lim_{x\to0}\frac{2x(1-\cos x)+(1+x^2)\sin x}{2x}\end{aligned}$$