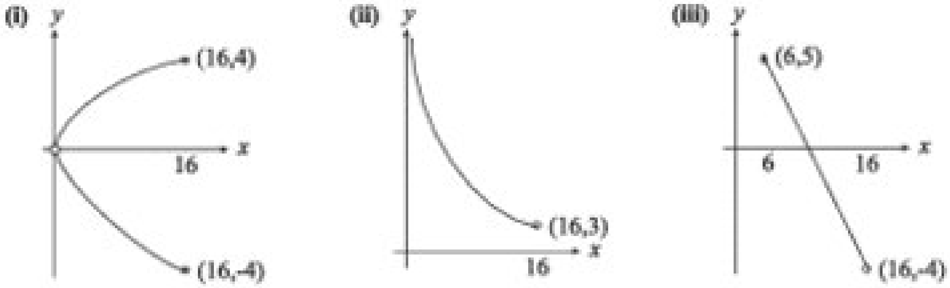
SOLUTIONS TO FINAL EXAM

1. [9 Marks] For each of the three diagrams (i), (ii), (iii) below :
(a) decide whether the diagram is the graph of a function with domain $(0, 16]$;
(b) if the answer to (a) is "no", explain why not, ie. justify your answer;
(c) if the answer to (a) is "yes", find the range of the given function.



Solution **(i)** This graph does not represent a function because for every $0 < x \le 16$ there are two different values of $f(x)$. A function must assign a **unique** value of $f(x)$ to each x in its domain.
(ii) This graph is the graph of a function. Its range consists of the projection $= [3, \infty)$ of the graph on the y-axis.
(iii) This graph does not represent a function because f does not assign values to numbers in $(0, 6)$. A function must assign a value to each number in its domain.

2. [12 Marks] Let $f(x) = |1 - x^2|$ with domain $\Re$. Let $g(x) = \sqrt{x+1}$ with domain $(-1, 2)$.

(a) Find $(f \circ g)(x)$. Simplify your answer and state the domain of $f \circ g$.

Solution **(a)** The domain of $f \circ g$ is the same as the domain $(-1, 2)$ of g.

$$(f \circ g)(x) = f(g(x)) = \left|1 - (\sqrt{x+1})^2\right| = |1 - (x+1)| = |-x| = |x|.$$

(b) Using interval notation, find the largest subset A of $(-1, 2)$ on which $(f \circ g)'(x)$ is defined. Find and simplify the value of this derivative on A.

Solution Since $(f \circ g)(x) = -x$ for $x \in (-1, 0)$ and $f(x) = x$ for $x \in (0, 2)$, $(f \circ g)'(x)$ exists at every $x \neq 0$. At $x = 0$, the graph has a cusp ($f'_+(0) = 1$ and $f'_-(1) = -1$). Hence $f'(0)$ does not exist. Thus $(f \circ g)'(x)$ exists for $x \in (-1, 0) \cup (0, 2)$. Moreover,

$$(f \circ g)'(x) = \begin{cases} -1 & \text{for } x \in (-1, 0) \\ +1 & \text{for } x \in (0, 2) \end{cases}$$

(c) State whether or not f has an inverse. If not, explain why not. If it does, find a simplified expression for f^{-1}, and state the domain of f^{-1}.

Solution f does not have an inverse because it is not one–to–one. For example, $f(-1/2) = f(1/2) = 3/4$.

(d) State whether or not g has an inverse. If not, explain why not. If it does, find a simplified expression for g^{-1}, and state the domain of g^{-1}.

Solution g has an inverse because it is one–to–one. The domain of g^{-1} equals the range of g which is $(0, \sqrt{3})$. Let $y = g^{-1}(x)$. Then

$$\begin{aligned} x &= g(y) = \sqrt{y+1} \\ x^2 &= y + 1 \\ y &= x^2 - 1 \end{aligned}$$

3. [15 Marks] Find the derivatives of the following functions:
(a) $y = x\cos(3x^2 + 5)$