DIVISIBLE MULTIPLICATIVE GROUPS OF FIELDS

GREG OMAN

Abstract. Some time ago, Laszlo Fuchs asked the following question: which abelian groups can be realized as the multiplicative group of (nonzero elements of) a field? The purpose of this note is to answer his query within the class of divisible abelian groups.

1. Introduction

In [7] (Problem 69), Laszlo Fuchs asks which abelian groups can be realized as the multiplicative group of (nonzero elements of) a field. Many decades later, this question is largely unsolved, though quite a few partial results have been obtained. We refer the reader to [1], [4]–[6], [8]–[12], [15]–[17], and [20]–[23] for a sampling of what is known on this question (and related results). As stated in the abstract, it is the purpose of this paper to solve Fuchs' problem for the class of divisible abelian groups.

We begin by recalling some definitions and results from the theory of divisible abelian groups to which we shall refer throughout the paper.

Definition 1. *An abelian group G (written additively) is divisible provided for every $g \in G$ and every positive integer n, there exists $h \in G$ with $nh = g$.*

The most natural nontrivial example of a divisible abelian group is the additive group $\mathbb{Q}$ of rational numbers. Another example is the direct limit of the cyclic groups $\mathbb{Z}/\langle p^n \rangle$ (p a prime). This group is the so-called *quasi-cyclic group of type p^∞*, denoted $C(p^\infty)$. Divisible abelian groups play a fundamental role in group theory. In particular, they are precisely the injective objects in the category of abelian groups. Moreover, their structure is well-understood:

Structure Theorem for Divisible Abelian Groups. Let G be an abelian group. Then G is divisible if and only if G is a direct sum of copies of $\mathbb{Q}$ and $C(p^\infty)$ for various primes p.

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