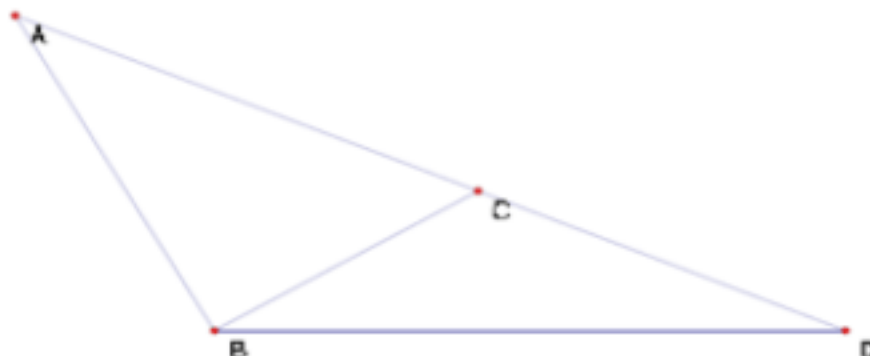# Proof of the Cube Root Construction

We begin with the following diagram:

The diagram is constructed with the following conditions:

- $\overline{AB} = \overline{CD} = 1$

- $\angle ABC = 90°$ and $\angle CBD = 30°$

The claim is that the segment $\overline{AC}$ has length $\sqrt[3]{2}$.

Proof: Just to make things easier, we will use x to represent the length of $\overline{AC}$. By the Pythagorean Theorem, we can say that the length of $\overline{BC}$ is $\sqrt{x^2 - 1}$. By addition, we can say the length of $\overline{AD}$ is $x + 1$.

The *Law of Sines* says that if ABC is a triangle with side lengths a, b, and c opposite the angles at A, B, and C respectively, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Using this on the small triangle BCD, we can say:

$$\frac{\sin \angle BDC}{\overline{BC}} = \frac{\sin \angle CBD}{\overline{CD}}$$

Substituting all we know, this says:

$$\frac{\sin \angle BDC}{\sqrt{x^2 - 1}} = \frac{\sin 30°}{1}$$

Square this to say $\sin^2 \angle BDC = (x^2 - 1)/4$.

Now we use the law of sines on the big triangle ABD:

$$\frac{\sin \angle ADB}{\overline{AB}} = \frac{\angle ABD}{\overline{AD}}$$

Substituting all we know, this says:

$$\frac{\sin \angle ADB}{1} = \frac{\sin 120°}{x + 1}$$