Chapter 7
Hamilton's Principle - Lagrangian and Hamiltonian Dynamics

Many interesting physics systems describe systems of particles on which many forces are acting. Some of these forces are immediately obvious to the person studying the system since they are externally applied. Other forces are not immediately obvious, and are applied by the external constraints imposed on the system. These forces are often difficult to quantify, but the effect of these forces is easy to describe. Trying to describe such a system in terms of Newton's equations of motion is often difficult since it requires us to specify the total force. In this Chapter we will see that describing such a system by applying Hamilton's principle will allow us to determine the equation of motion for system for which we would not be able to derive these equations easily on the basis of Newton's laws. We should stress however, that Hamilton's principle does not provide us with a new physical theory, but it allows us to describe the existing theories in a new and elegant framework.

Hamilton's Principle

The evolution of many physical systems involves the minimization of certain physical quantities. We already have encountered an example of such a system, namely the case of refraction where light will propagate in such a way that the total time of flight is minimized. This same principle can be used to explain the law of reflection: the angle of incidence is equal to the angle of reflection.

The minimization approach to physics was formalized in detail by Hamilton, and resulted in Hamilton's Principle which states:

> " *Of all the possible paths along which a dynamical system may more from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.* "

We can express this principle in terms of the calculus of variations:

$$\delta\int_{t_1}^{t_2}(T-U)\,dt=0$$

The quantity T - U is called the Lagrangian L.

Consider first a single particle, moving in a conservative force field. For such a particle, the kinetic energy T will just be a function of the velocity of the particle, and the potential energy will just be a function of the position of the particle. The Lagrangian is thus also a function of the position and the velocity of the particle. Hamilton's theorem states that we need to minimize the Lagrangian and thus require that